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NON-ARCHIMEDEAN DIFFERENTIATION

by

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Introduction

This note is a collection of the most important results of a lengthy paper that will appear as a Report, Mathematisch Instituut, Katholieke Universiteit, Nijmegen. There the interested reader may find all the proofs that are missing here, and also a bibliography. Because of the rather complicated formulas and computations that are involved it seems wise to restrict oneself first to K -valued functions of one single variable. However, a lot of the results can without any problem be carried over to E -valued functions of one variable, where E is a K -Banach space. A generalization to functions: $K^n \rightarrow K^m$ will be less obvious, although it seems clear how to define C^k -functions in that case. (For example, in order that $f : K^2 \rightarrow K$ is C^1 one should require (see 3.1) that the difference quotients

$$(x_1, x_2, y) \mapsto \frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2}, \quad (x, y_1, y_2) \mapsto \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}$$

can be extended to continuous functions on K^3 . If we take again difference quotients we get four functions of four variables, required to be continuous in order that f be in C^2 (see 6.1). It then follows very easily that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for $f \in C^2$.)

Throughout this note, K will always be a complete non-archimedean valued field, and X a non-empty subset of K , without isolated points. We study differentiability properties of functions $f : X \rightarrow K$. Besides the analytic functions we have other examples of differentiable

functions such as the locally constant functions (they have derivative zero everywhere). The function $\sum a_n p^n \rightarrow \sum a_n p^{n!}$, defined on \mathbb{Z}_p is an example of an injective function with zero derivative and which is in Lip_α for every $\alpha > 0$. The function $f : \mathbb{Z}_p \rightarrow \mathbb{Q}_p$ defined via $f(x) = x - p^{2n}$ if $|x - p^n| < p^{-2n}$ and $f(x) = x$ elsewhere has derivative 1 everywhere, but for all $n \in \mathbb{N}$ $f(p^n) = f(p^n - p^{2n}) = p^n - p^{2n}$, hence f is not locally injective at 0.

The above examples show that we do not have a mean value theorem or a local invertibility theorem. We now state the results going from "bad" to "smooth" functions.

1. Nowhere differentiable functions

Let $BC(X)$ be the algebra of the bounded continuous functions: $X \rightarrow K$, normed by the sup norm $\| \cdot \|_\infty$. We have, analogous to the classical case:

THEOREM 1.1. The collection of those $f \in BC(X)$ that are somewhere differentiable is of first category in $BC(X)$ (in the sense of Baire).

In contrast to the theory of functions on the real line we have

THEOREM 1.2. Let X be open in K , and let $f : X \rightarrow K$ be a bounded uniformly continuous function, and let $\epsilon > 0$. Then there exists a nowhere differentiable $g : X \rightarrow K$ such that g has bounded difference quotients, and such that $\|f - g\|_\infty < \epsilon$.

2. Differentiability as such

Contrary to the classical case we have a nice criterion for a

function to possess an antiderivative:

THEOREM 2.1. Let $f : X \rightarrow K$. Then f has an antiderivative if and only if f is of Baire class one. (i.e., f is the pointwise limit of a sequence of continuous functions).

Further we have Sard-type theorems. If K is a local field then $Y \subset X$ is called a nullset if it has measure zero in the sense of the (real) Haar measure on K .

THEOREM 2.2. Let K be a local field and let $f : X \rightarrow K$ be differentiable. Then we have:

- (1) If $Y \subset X$ is a nullset then $f(Y)$ is a nullset ("f has property (N))"
- (2) $\{f(x) : f'(x) = 0\}$ is a nullset.

COROLLARY 2.3. If $f : X \rightarrow K$ is differentiable, $f' = 0$ almost everywhere, then $f(X)$ is a nullset.

3. Continuously differentiable functions

If we want the local invertibility theorem to hold for C^1 -functions we have to take a definition of a C^1 -function, stronger than just " f is differentiable and f is continuous". For $f : X \rightarrow K$, define

$$\phi_1 f(x, y) = \frac{f(x) - f(y)}{x - y} \quad (x, y \in X, x \neq y).$$

DEFINITION 3.1. $f : X \rightarrow K$ is in $C^1(X)$ if $\phi_1 f$ can (uniquely) be extended to a continuous function $\overline{\phi_1 f}$ on $X \times X$.

(Notice that for a real valued function f defined on an interval

the continuity of f' already guarantees the existence of a continuous $\bar{\phi}_1 f$.

THEOREM 3.2. Let $f \in C^1(X)$ and let $a \in X$.

- (a) If $f'(a) \neq 0$ then f is locally invertible at a . (In fact, $(f'(a))^{-1}f$ is an isometry locally at a).
- (b) If X is open in K and if $f' \neq 0$ everywhere on X then f is an open mapping.

Let $BC^1(X) = \{f \in C^1(X) : \|f\|_1 := \|f\|_\infty \vee \|\bar{\phi}_1 f\|_\infty\}$. Then $BC^1(X)$ is a Banach space with respect to $\|\cdot\|_1$. We may put a locally convex topology on $C^1(X)$ via the defining seminorms $\|\cdot\|_{1,C}$ where C runs through the compact subsets of X :

$$\|f\|_{1,C} = \sup_{x \in C} |f(x)| \vee \sup_{\substack{x \in C \\ y \in C}} |\bar{\phi}_1 f(x, y)| \quad (f \in C^1(X)).$$

Let $N^1(X) = \{f \in C^1(X) : f' = 0\}$ and $BN^1(X) = \{f \in BC^1(X) : f' = 0\}$. Then $N^1(X)$ is closed in $C^1(X)$, $BN^1(X)$ is closed in $BC^1(X)$.

THEOREM 3.3. The locally linear functions (in $BC^1(X)$) form a dense subset of $C^1(X)$ (of $BC^1(X)$).

The locally constant functions (in $BC^1(X)$) form a dense subset of $N^1(X)$ (of $BN^1(X)$).

THEOREM 3.4. If either X is compact or K has discrete valuation then $BC^1(X)$ has an orthonormal base (in the sense of the norm).
If X is not compact and K has dense valuation then, for any $\alpha > 0$, $BC^1(X)$ has no α -orthogonal base.

Let us choose real numbers $1 > r_1 > r_2 > \dots$ with $\lim r_n = 0$, and, for each n , let R_n be a full set of representatives of the equivalence relation (in X): $x \sim y$ if $|x-y| < r_n$. We can arrange that $R_1 \subset R_2 \subset \dots$. For each $x \in X$, $n \in \mathbb{N}$, let $x_n \in X$ be determined by: $|x_n - x| < r_n$, $x_n \in R_n$. For a continuous $f : X \rightarrow K$ set

$$(Pf)(x) = \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X)$$

THEOREM 3.5. The map P defined above is a continuous linear map:

$C(X) \rightarrow C^1(X)$ and its restriction to $BC(X)$ is an isometry: $BC(X) \rightarrow BC^1(X)$. P is an antiderivation map i.e., $(Pf)' = f$ for each $f \in C(X)$.

COROLLARY 3.6. Every continuous function has a C^1 -antiderivative.

In fact, by passing through the quotient, differentiation yields a map $\rho : BC^1(X)/BN^1(X) \rightarrow BC(X)$ which is a surjective isometry. Moreover, $BN^1(X)$ has an orthogonal complement $(\text{im } P)$ in $BC^1(X)$.

4. $C^1(X)$ for compact X

(Throughout section 4, X is compact). The set $\{|x-y| : x, y \in X\}$ is bounded and has only 0 as an accumulation point, hence it can be written as $\{r_1, r_2, \dots\} \cup \{0\}$, where $r_1 > r_2 > \dots$ and $\lim r_n = 0$. Let $r_0 = \infty$. For each i , let R_i be a full set of representatives in X of the equivalence relation " $x \sim y$ if $|x-y| < r_i$ " such that $R_0 \subset R_1 \subset \dots$. Then R_i is finite for each i and R_0 consists only of one single point a_0 . Let $R = \bigcup_i R_i$ and define $v : R \rightarrow \{0, 1, 2, \dots\}$ as follows. For $a \in R$ let $v(a)$ be the nonnegative integer m for which $a \in R_m \setminus R_{m-1}$ ($R_{-1} = \emptyset$ by definition). For each $a \in R$ let

$$B_a = \{x \in X : |x-a| < r_{v(a)}\},$$

and let e_a be the K -valued characteristic function of B_a . Further, we define

$$a \triangleleft b \text{ iff } b \in B_a \quad (a, b \in R)$$

Then we have

LEMMA 4.1. (R, \triangleleft) is a partially ordered set with a smallest element a_0 . For each $a \in R$, the set $\{x \in R : x \triangleleft a\}$ is finite and linearly ordered by \triangleleft .

Define for $a \in R$, $a \neq a_0$: $a_- = \max \{x \in R : x \neq a, x \triangleleft a\}$. Then

THEOREM 4.2. The set $\{e_a : a \in R\}$ forms an orthonormal base of $C(X)$.

Let $f \in C(X)$ and $f = \sum \lambda_a e_a$ for some $\lambda_a \in K$. Then

$$\lambda_{a_0} = f(a_0) \text{ and for } a \neq a_0: \lambda_a = f(a) - f(a_-).$$

The set $\{e_a : a \in R\} \cup \{pe_a : a \in R\}$ (p as in 3.5) forms

an orthogonal base of $C^1(X)$. Let $f \in C^1(X)$, $f = \sum \lambda_n e_a + \sum \mu_b pe_b$

$(\lambda_a, \mu_b \in K)$ in the $|||_1$ -norm. Then $\lambda_{a_0} = f(a_0)$, $\mu_{a_0} = f'(a_0)$

and for $a \neq a_0$:

$$\lambda_a = f(a) - f(a_-) - (a - a_-)f'(a_-)$$

$$\mu_a = f'(a) - f'(a_-).$$

5. Uniform differentiability

There seem to be two natural notions of "uniform differentiability".

Let $f \in C^1(X)$. f is called uniformly differentiable if $\lim_{x \rightarrow y} \phi_1 f(x, y) = f'(y)$ uniformly in y . f is called strongly uniformly differentiable if $\bar{\phi}_1 f$ is uniformly continuous.

If X is compact both notions are the same and coincide with "continuous differentiable".

THEOREM 5.1. Let $f : X \rightarrow K$ be (strongly) uniformly differentiable. Then f has a unique continuous extension $\bar{f} : \bar{X} \rightarrow K$ (\bar{X} is the closure of X in K).

This \bar{f} is (strongly) uniformly differentiable.

THEOREM 5.2. Let $f : X \rightarrow K$ be uniformly differentiable. Then each of the following properties implies strong uniform differentiability of f :

- (a) $\phi_1 f$ is bounded.
- (b) Both f and f' are bounded
- (c) X is "nice" and f is bounded.

(X is called "nice" if for each $r > 0$ there is $s > 0$ such that for every $x \in X$ there is $y \in X$ such that $s \leq |x-y| \leq r$).

The theorems 3.3, 3.5., 3.6. each have an analogon for uniformly differentiable functions.

6. C^n -functions

For $n \in \mathbb{IN}$, let $\nabla^n X = \{(x_1, \dots, x_n) \in X^n : i \neq j \Rightarrow x_i \neq x_j\}$. For $f : X \rightarrow K$ we define the n^{th} difference quotient $\phi_n f : \nabla^{n+1} X \rightarrow K$ inductively as follows $\phi_0 f = f$ and for $(x_1, \dots, x_{n+1}) \in \nabla^{n+1} X$:

$$\phi_n f(x_1, \dots, x_{n+1}) = (x_1 - x_2)^{-1} (\phi_{n-1} f(x_1, x_3, \dots, x_{n+1}) - \phi_{n-1} f(x_2, x_3, \dots, x_{n+1})).$$

Since $\nabla^n X$ is dense in X^n for each n the following definition makes sense.

DEFINITION 6.1. Let $f : X \rightarrow K$, $n \in \mathbb{IN} \cup \{0\}$. We say that $f \in C^n(X)$ if

$\phi_n f$ can be extended to a continuous function $\bar{\phi}_n f : X^{n+1} \rightarrow K$.

We say that $f \in B\Delta^n(X)$ if $\phi_0 f, \dots, \phi_n f$ are bounded functions.

For $f \in B\Delta^n(X)$ set

$$||f||_n = \max_{0 \leq i \leq n} ||\phi_i f||_\infty.$$

Let $BC^n(X) = B\Delta^n(X) \cap C^n(X)$, $C^\infty(X) = \bigcap_{n=1}^{\infty} C^n(X)$,

$$BC^\infty(X) = \bigcap_{n=1}^{\infty} BC^n(X).$$

THEOREM 6.2. $C^1(X) \supset C^2(X) \supset \dots$

$$B\Delta^1(X) \supset BC^1(X) \supset B\Delta^2(X) \supset BC^2(X) \supset \dots$$

$B\Delta^n(X)$ is a Banach space with respect to $|| \cdot ||_n$ and

$BC^n(X)$ is closed in $B\Delta^n(X)$.

For $f \in C^n(X)$ ($n \geq 1$) and $0 \leq j \leq n$ we define the j^{th} Hasse derivative of f by

$$D_j f(x) = \bar{\phi}_j f(x, x, \dots, x) \quad (x \in X).$$

THEOREM 6.3. Let $f \in C^n(X)$. Then for $0 \leq j \leq n$ we have $D_i f \in C^{n-i}(X)$ and if $i+j \leq n$

$$D_i D_j f = \binom{i+j}{i} D_{i+j} f$$

f is n times differentiable in the ordinary sense and
for $0 \leq i \leq n$ we have

$$f^{(i)} = i! D_i f.$$

$f : X \rightarrow K$ is called a spline function of degree $\leq n$ if for every $a \in X$ there is a neighbourhood U of a such that $f|_{U \cap X}$ is a polynomial function of degree $\leq n$. Spline functions are in $C^\infty(X)$.

THEOREM 6.4. Let $f \in C^n(X)$ and $\epsilon > 0$. Then there is a spline function
 g of degree $\leq n$ such that $f-g \in BC^n(X)$, $||f-g||_n < \epsilon$. If
 $D_i f = D_{i+1} f = \dots = D_n f = 0$ for some $i \in \{1, \dots, n\}$ then
 g can be chosen to be of degree $\leq i-1$.

THEOREM 6.5. (Local invertibility). Let $f \in C^n(X)$ and $f'(a) \neq 0$ for
some $a \in X$. Then there is a neighbourhood U of a ($a \in U \subset X$)
such that $f : U \rightarrow f(U)$ is a bijection, and such that the
local inverse: $f(U) \rightarrow U$ is in $C^n(f(U))$.

THEOREM 6.6. (Taylor formula). Let $f \in C^n(X)$. Then for all $x, y \in X$:

$$f(x) = f(y) + (x-y)D_1f(y) + \dots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^n \bar{D}_n f(x, y, \dots, y).$$

The above result leads to another possible notion of "n-times continuously differentiable":

DEFINITION 6.7. Let $f : X \rightarrow K$, $n \in \mathbb{N}$. We say that $f \in C^n(X)$ if there
exist functions $D_1f, \dots, D_{n-1}f : X \rightarrow K$ and a continuous
 $R_n f : X^2 \rightarrow K$ such that for all $x, y \in X$

$$f(x) = f(y) + (x-y)D_1f(y) + \dots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^n R_n f(x, y).$$

(It follows that the $D_i f$, $R_n f$ are uniquely determined and continuous.
 Further we have $C^1(X) \supset C^2(X) \supset \dots$). It is easy to show that $C^i(X) = C^i(X)$
 for $i = 1, 2$. Also $C^n(X) \subset C^n(X)$ for all n , by 6.6. But we have

EXAMPLE 6.8. Let $X = \{\sum a_n p^{n!} \in \mathbb{Z}_p : a_n \in \{0, 1\}\}$, and let $f : X \rightarrow K$ be
defined via

$$f(\sum a_n p^{n!}) = \sum a_n p^{3n!}.$$

Then $f \in C^n(X)$ for each n , and $D_i f = 0$ for $i = 1, 2, 4, 5, \dots$
and $D_3 f = 1$. On the other hand, $f \notin C^3(X)$.

Let $C > 0$ and $\{x_1, \dots, x_n\}$ a set of n distinct points in X . We call
 $\{x_1, \dots, x_n\}$ a C-polygon if for all $i, j, k, l \in \{1, \dots, n\}$, $k \neq l$:

$$\frac{|x_i - x_j|}{|x_k - x_1|} \leq C.$$

DEFINITION 6.9. Let $n \in \mathbb{N}$. We say that X has locally property B_n if
for each $a \in X$ there is $\delta > 0$ and $C > 0$ such that for
all $x_1, x_2 \in X$, $x_1 \neq x_2$, $|x_1 - a| < \delta$, $|x_2 - a| < \delta$ there
exist $x_3, \dots, x_n \in X$ such that $\{x_1, x_2, \dots, x_n\}$ is a C -
polygon. (By definition, every X has locally property
 B_1 and B_2).

We say that X has globally property B_n if there exists
 $C > 0$ such that for all $x_1, x_2 \in X$, $x_1 \neq x_2$, there exist
 $x_3, \dots, x_n \in X$ such that $\{x_1, x_2, \dots, x_n\}$ is a C -polygon.
(Every X has globally property B_1 and B_2).

For example, a ball in K has globally property B_n , for each n . Every
open (non-empty) subset of K has locally property B_n , for each n .

Let us call $BC^n(X) = \{f \in C^n(X) : ||f||_n^\vee < \infty\}$, where, by definition,

$$||f||_n^\vee = \max(||f||_\infty, ||\mathcal{D}_1 f||_\infty, \dots, ||\mathcal{D}_{n-1} f||_\infty, ||R_n f||_\infty)$$

(see 6.7). It is very easy to show that $BC^n(X)$ is a Banach space with
respect to $|| \cdot ||_n^\vee$. The main theorem:

THEOREM 6.10. If X has locally property B_n , then $C^n(X) = BC^n(X)$.

Let X have globally property B_n ($n \geq 2$) in the sense
that every two-point set can be extended to a C -polygon.
Then $BC^n(X) = BC^n(X)$ and

$$|| \cdot ||_n^\vee \leq || \cdot ||_n \leq C^{2(n-2)} || \cdot ||_n^\vee.$$

(In general we have for $f \in BC^n(X)$: $\|f\|_n = \max_{0 \leq i \leq n} \|D_i f\|_{n-i}^v$).

As in 3.5. we want to find an antiderivation map: $C^{n-1}(X) \rightarrow C^n(X)$. We cannot use the map P of 3.5. since one can prove: if $f \in C^1(X)$ then $Pf \in C^2(X)$ if and only if $f' = 0$. Further, if the characteristic of K equals $p \neq 0$ it is easy to see that not every C^{p-1} -function has a C^p -antiderivative.

THEOREM 6.11. Let the characteristic of K be zero and let $r_1 > r_2 \dots$ as in 3.5 but such that $r_m \leq \rho r_{m+1}$ for all m , some $\rho > 0$. For $f \in C^{n-1}(X)$, set

$$P_n f(x) = \sum_{k=1}^{\infty} \sum_{i=1}^n \frac{1}{i!} (x_{k+1} - x_k)^i D_{i-1} f(x_k) \quad (x \in X)$$

Then $P_n f \in C^n(X)$ and $(P_n f)' = f$. If $f \in BC^{n-1}(X)$, then $Pf \in BC^n(X)$ and

$$\|P_n f\|_n \leq c_n \|f\|_{n-1}$$

where

$$c_n = \max_{1 \leq i \leq n} \frac{1}{i!} \cdot \rho^n.$$

It follows that the map $Q_n = n! P_n P_{n-1} \dots P_1$ sends $C(X)$ into $C^n(X)$, $D_n Q_n$ is the identity on $C(X)$. A computation yields

$$Q_n = \sum_{i=1}^n (-1)^{i+1} \binom{n}{i} M^{n-i} S_i,$$

where M is the multiplication with x ($(Mf)(x) = xf(x)$ for $f \in C(X)$)

and where

$$S_i f(x) = \sum_{k=1}^{\infty} f(x_k) (x_{k+1}^i - x_k^i) \quad (f \in C(X), x \in X)$$

Note: Theorem 3.4 is also true if we replace $BC^1(X)$ by $BC^n(X)$ ($n \in \mathbb{IN}$).

7. C^∞ -functions

Spline functions, analytic functions are in $C^\infty(X)$.

Let $f \in C^\infty(X)$ and let $f(a) = 0$ for some $a \in X$. Then $f(x) = (x-a)g(x)$ ($x \in X$) where $g \in C^\infty(X)$.

A derivation on $C^\infty(X)$ is a linear map $\phi : C^\infty(X) \rightarrow C^\infty(X)$ such that

$$\phi(fg) = \phi(f)g + f\phi(g) \quad (f, g \in C^\infty(X)).$$

Any derivation ϕ has the form $g \frac{d}{dx}$, where $g \in C^\infty(X)$.

THEOREM 7.1. Let $Y \subset K$ be a closed subset of K . Then there is $f \in C^\infty(K)$ (with $f' = 0$ everywhere) such that Y is the set of zeros of f .

THEOREM 7.2. Let $\lambda_0, \lambda_1, \dots$ be any sequence in K . Then there exists an $f \in C^\infty(K)$ such that $D_i f(0) = \lambda_i$ for all i .

Open problem: Let the characteristic of K be zero. Does every $f \in C^\infty(X)$ have a C^∞ -antiderivative?